

UNIFORM LIMIT THEOREMS UNDER RANDOM TRUNCATION

V. FAKOOR¹ R. ZAMINI²

¹*Department of Statistics, Faculty of Mathematical Sciences,
Ferdowsi University of Mashhad, Iran.*

²*Department of Mathematics and Computer Science, Kharazmi University, Tehran, Iran.*

ABSTRACT. In this paper we study uniform versions of two limit theorems in random left truncation model (RLTM). The law of large numbers (LLN) and the central limit theorem (CLT) have been obtained under the bracketing entropy conditions in this setting. The uniform LLN and the uniform CLT of the present paper extend the one dimensional LLN and the one dimensional CLT under RLTM respectively.

1. Introduction and Preliminaries

Limit theorems have always been of central importance in probability theory. Two most important of limits theorems constitute: the LLN and the CLT. These topics have their own importance in the classical probability theory as well applications in statistical inference. In recent years, under the topic of empirical process theory, authors have become interested in the uniform analogue of the two theorems: the uniform LLN of Glivenko-Cantelli type and the uniform CLT for Donsker type.

Let X_1, \dots, X_n be a sequence of random variables defined on a probability space (Ω, \mathcal{T}, P) with common law distribution \mathbb{P} on \mathbb{R} . We denote by $\mathbb{P}_n = n^{-1} \sum_{i=1}^n \delta_{(X_i)}$ the usual empirical measure, where δ_x is a Kronecker delta. Let \mathcal{F} be a set of measurable real valued functions on \mathbb{R} . The uniform version of the LLN states that

$$(1.1) \quad \sup_{\varphi \in \mathcal{F}} (\mathbb{P}_n - \mathbb{P})\varphi \rightarrow 0 \quad a.s.,$$

where $\mathbb{P}\varphi = \int \varphi d\mathbb{P}$. A class \mathcal{F} , for which (1.1) holds, is called a *Glivenko-Cantelli class*. DeHardt [2] obtained the uniform LLN for the sequence of independent and identically distributed (i.i.d.) random variables under bracketing entropy (see for example Van der Vaart and Wellner [10]). DeHardt's result states that if \mathcal{F} has a bracketing entropy then (1.1) is obtained.

The \mathcal{F} -indexed empirical process is given by

$$(1.2) \quad \varphi \mapsto \sqrt{n}(\mathbb{P}_n - \mathbb{P})\varphi \quad \varphi \in \mathcal{F}.$$

This process can be viewed as a map into $\ell^\infty(\mathcal{F})$, where $\ell^\infty(\mathcal{F})$ denotes the Banach space of bounded real-valued functions ψ on \mathcal{F} , normed by $\|\psi\|_{\mathcal{F}} := \sup_{\varphi \in \mathcal{F}} |\psi(\varphi)|$. The processes $\varphi \mapsto \sqrt{n}(\mathbb{P}_n - \mathbb{P})\varphi, \varphi \in \mathcal{F}$ converge in law to a Gaussian process in $\ell^\infty(\mathcal{F})$, called

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the \mathbb{P} -Brownian bridge indexed by \mathcal{F} . A class \mathcal{F} for which this is the true, is named a \mathbb{P} -Donsker class (see Van der Vaart and Wellner [10]).

The study of the asymptotic behaviors of the process (1.1) is a central topic in empirical process theory, and it is well known that this behavior depends on the complexity or "entropy" of \mathcal{F} .

Ossiander [7] developed the uniform CLT for the sequence of i.i.d. random variables under certain metric integrability conditions. The Ossiander result states that under an integrability condition on the metric entropy with bracketing in $L^2(\mathbb{P})$, $\sqrt{n}(\mathbb{P}_n - \mathbb{P})$, as random elements in $\ell^\infty(\mathcal{F})$, converges in law to a mean zero Gaussian process $\{W(\varphi) : \varphi \in \mathcal{F}\}$ with the covariance structure $EW(\varphi_1)W(\varphi_2) = E\varphi_1(X)\varphi_2(X)$.

There is lack of works on uniform limit theorems for a RLTM in the literature. The goal of the paper is to prove two fundamental theorems of uniform LLN and CLT . Here we present the uniform LLN of Glivenko-Cantelli type and the uniform CLT for Donsker type in the RLTM which postpone to section 2.

Consider that there is a finite population \mathcal{P} whose size is large, deterministic and is denoted by N . Each element of \mathcal{P} contains two independent random variables denoted by Y and T with distribution functions, respectively shown by F and G . Y is the variable of interest and T is the left truncation random variable. So until now we have N i.i.d. random variables $\{(Y_i, T_i); 1 \leq i \leq N\}$. Suppose that (Y, T) is observed if $Y \geq T$, otherwise we have no information about them. We show the observable random variables by $\{(Y_i, T_i); 1 \leq i \leq n\}$ with $Y_i \geq T_i$. As a direct result n is a binomial random variable, with sample size N and success probability $\alpha = P(Y \geq T)$.

The conditional distribution of (Y, T) given $Y \geq T$ is denoted by H^* ,

$$\begin{aligned} H^*(y, t) &= P(Y \leq y, T \leq t | Y \geq T) \\ (1.3) \quad &= \alpha^{-1} \int_{-\infty}^y G(t \wedge u) dF(u), \end{aligned}$$

Marginal distribution functions of Y and T are given by

$$F^*(y) = H^*(y, \infty) = \alpha^{-1} \int_{-\infty}^y G(u) dF(u),$$

and

$$G^*(t) = H^*(\infty, t) = \alpha^{-1} \int_{-\infty}^t (1 - F(u)) dG(u).$$

The corresponding empirical distributions are defined by

$$F_n^*(y) = \frac{1}{n} \sum_{i=1}^n I(Y_i \leq y), \quad G_n^*(t) = \frac{1}{n} \sum_{i=1}^n I(T_i \leq t),$$

where $I(A)$ is the indicator function of A .

Random truncation restricts the observation range of X and Y . Only $F_0(x) = P(X \leq x | X \geq a_G)$ and $G_0(y) = P(Y \leq y | Y \leq b_F)$ can be estimated, where

$$a_F := \inf\{x : F(x) > 0\} \quad \text{and} \quad b_F := \sup\{x : F(x) < 1\}$$

are the lower and upper boundaries of the support of the distribution of X . Let a_G and b_G be similarly defined. If $a_G \leq a_F$, then $F = F_0$.

The nonparametric maximum likelihood estimator of F was first derived by Lynden-Bell [6], which we refer to it by $F_n(\cdot)$. If there are no ties in the data, it is given by

$$(1.4) \quad F_n(y) = 1 - \prod_{i: Y_i \leq y} \left[\frac{nC_n(Y_i) - 1}{nC_n(Y_i)} \right],$$

in which

$$C_n(y) = G_n^*(y) - F_n^*(y^-) = \frac{1}{n} \sum_{i=1}^n I_{\{T_i \leq y \leq Y_i\}}, \quad y \in \mathbb{R},$$

where is the empirical estimator of

$$(1.5) \quad C(y) := G^*(y) - F^*(y) = \alpha^{-1} G(y) (1 - F(y^-)), \quad y \in \mathbb{R}.$$

Left limit $\lim_{y \uparrow s} g(y)$ is denoted by $g(s^-)$.

The following definitions, are conveniently collected in Van der Vaart and Wellner [10], so we follow their notation. First, we define entropy with bracketing. Let $(\mathcal{F}, \|\cdot\|_p)$ be a subset of a normed space of $(L^p(\mathbb{P}), \|\cdot\|_p)$ where

$$(\|\varphi\|_p)^p = \int |\varphi|^p d\mathbb{P}.$$

Definition 1. Given two functions l and u in $L^p(\mathbb{P})$, the bracket $[l, u]$ is the set of all functions φ with $l \leq \varphi \leq u$. An ϵ -bracket in $L^p(\mathbb{P})$ is a bracket $[l, u]$ with $\|u - l\|_p < \epsilon$. The bracketing number $N_{[\cdot]}(\epsilon, \mathcal{F}, \|\cdot\|_p)$ is the minimum number of ϵ -brackets needed to cover \mathcal{F} . The entropy with bracketing is the logarithm of the bracketing number.

The notion of entropy with bracketing has been introduced by Dudley [3] and the importance of $L^2(\mathbb{P})$ -entropy with bracketing has been pointed out by Ossiander [7]. In the rest of the paper, whenever unambiguous we write $N_{[\cdot]}(\epsilon)$ instead of $N_{[\cdot]}(\epsilon, \mathcal{F}, \|\cdot\|_p)$.

Definition 2. The bracketing entropy integral of a class of functions \mathcal{F} is defined by

$$(1.6) \quad J(\delta) := J_{[\cdot]}(\delta, \mathcal{F}, \|\cdot\|_p) = \int_0^\delta \sqrt{\log N_{[\cdot]}(\epsilon, \mathcal{F}, \|\cdot\|_p)} d\epsilon,$$

where $N_{[\cdot]}(\cdot)$ are the bracketing numbers of \mathcal{F} with respect to the norm $\|\cdot\|_p$.

Definition 3. A measurable function Φ is an envelope function for \mathcal{F} if $|\varphi(x)| \leq \Phi(x)$ for all $\varphi \in \mathcal{F}$ and all x . If $\sup_{\varphi \in \mathcal{F}} |\varphi(x)|$ is measurable, it will be called the envelope function of \mathcal{F} .

Definition 4. A sequence of $\ell^\infty(\mathcal{F})$ -valued random functions $\{Y_n\}$ converges almost surely to a constant c if $P^*(\sup_{\varphi \in \mathcal{F}} Y_n(\varphi) \rightarrow c) = P(\sup_{\varphi \in \mathcal{F}} Y_n(\varphi)^* \rightarrow c) = 1$. Here P^* denotes the outer probability, and $\sup_{\varphi \in \mathcal{F}} Y_n(\varphi)^*$ is the measurable cover function of $\sup_{\varphi \in \mathcal{F}} Y_n(\varphi)$.

We use the following definition of weak convergence which is originally due to Hoffman-Jørgensen [5].

Definition 5. A sequence of $\ell^\infty(\mathcal{F})$ -valued random functions $\{Y_n\}$ converges in law to a $\ell^\infty(\mathcal{F})$ -valued random function $\{Y\}$ whose law concentrates on a separable subset of $\ell^\infty(\mathcal{F})$ if

$$Eg(Y) = \lim_{n \rightarrow \infty} E^*g(Y_n) \quad \forall g \in U(\ell^\infty(\mathcal{F}), \|\cdot\|_{\mathcal{F}}),$$

where $U(\ell^\infty(\mathcal{F}), \|\cdot\|_{\mathcal{F}})$ is the set of all bounded, uniformly continuous function from $(\ell^\infty(\mathcal{F}), \|\cdot\|_{\mathcal{F}})$ into \mathbb{R} . Here E^* denotes the upper expectation with respect to the outer probability P^* . We denote this convergence by $Y_n \Rightarrow Y$.

The layout of this paper is as follows. In Section 2, we obtain our main theorems and results. In order to prove the main theorems we need the some auxiliary results, which are contained in Section 3.

2. Main results

2.1. Uniform LLN. In this subsection, we assume that \mathcal{F} is a class of real-valued measurable functions on \mathbb{R} such that $\mathcal{F} \subseteq L^1(F) := \{\varphi : \int |\varphi(x)|F(dx) < \infty\}$. The below assumption is imposed throughout this subsection to achieve Glivenko-Cantelli type theorem for $W_n(\varphi) := \int \varphi d(F_n - F)$.

Assumption A F and G are continuous with $a_G < b_F$.

Theorem 1. Let $N_{[\cdot]}(\epsilon, \mathcal{F}, L^1(F)) < \infty$, for every $\epsilon > 0$, then under Assumption **A** we have

$$(2.1) \quad \sup_{\varphi \in \mathcal{F}} \left| \int \varphi d(F_n - F) \right| \longrightarrow 0 \quad a.s.$$

as $n \rightarrow \infty$.

In order to start the proof of Theorem 1, we shall state the following lemma.

Lemma 1. Under **A**, for any measurable φ , one can write

$$\lim_{n \rightarrow \infty} \int \varphi dF_n = \int \varphi dF \quad a.s.$$

Proof. By using relation of $\varphi = \varphi^+ - \varphi^-$ in Theorem 4.3. of He and Yang [4] we can obtain the result. \square

Proof of Theorem 1. Let $\epsilon > 0$ is given. By definition of the bracketing number, we can find finitely many ϵ -brackets $[l_i, u_i]$ whose union contains \mathcal{F} and such that $\int (u_i - l_i)dF < \epsilon$ for every $i = 1, \dots, N_{[\cdot]}(\epsilon)$. Then, for every $\varphi \in \mathcal{F}$, there is bracket such that

$$\begin{aligned} W_n(\varphi) &= \int \varphi dF_n - \int \varphi dF \\ &\leq \int u_i d(F_n - F) + \int (u_i - l_i)dF \end{aligned}$$

Therefore,

$$\sup_{\varphi \in \mathcal{F}} W_n(\varphi) \leq \max_{1 \leq i \leq N_{[\cdot]}(\epsilon)} \int u_i d(F_n - F) + \epsilon$$

Thus, by Lemma 1

$$\limsup_{n \rightarrow \infty} \sup_{\varphi \in \mathcal{F}} W_n(\varphi) \leq \epsilon \quad a.s.$$

Combination with a similar argument for $\inf_{\varphi \in \mathcal{F}} W_n(\varphi)$ yields that

$$\limsup_{n \rightarrow \infty} \sup_{\varphi \in \mathcal{F}} |W_n(\varphi)|^* \leq \epsilon \quad a.s.$$

for every $\epsilon > 0$. Take a sequence $\epsilon_m \downarrow 0$ to see that \limsup must actually be zero almost surely. \square

Corollary 1. *Let $\varphi_0 : \mathbb{R} \rightarrow \mathbb{R}$ be a measurable function such that $\int |\varphi_0| dF < \infty$. Then,*

$$(2.2) \quad \sup_{t \in \mathbb{R}} \left| \int_{-\infty}^t \varphi_0(x)(F_n - F)(dx) \right| \longrightarrow 0 \quad a.s.$$

as $n \rightarrow \infty$.

Proof. Apply Theorem 1 to $\mathcal{F} = \{\varphi_0 \cdot I(-\infty, t] : t \in \mathbb{R}\}$. \square

2.2. Uniform CLT. In this subsection we give a CLT for \mathcal{F} -indexed empirical processes

$$G_n(\varphi) := \sqrt{n} \int \varphi d(F_n - F), \quad \varphi \in \mathcal{F}.$$

We assume that \mathcal{F} is a class of real-valued measurable functions on \mathbb{R} such that $\mathcal{F} \subseteq L^2(F) := \{\varphi : \int \varphi^2(x)F(dx) < \infty\}$. The below assumption is imposed throughout this subsection to achieve Donsker type theorem for $G_n(\varphi)$.

Assumption B. F is continuous with $a_G < a_F$.

Theorem 2. *Let \mathcal{F} be a class of functions with $J(1) < \infty$, satisfying Assumption B. Then $G_n \Rightarrow W$ as elements of $B(\mathcal{F})$ where $\{W(\varphi) : \varphi \in \mathcal{F}\}$ is a Gaussian process with the mean $EW(\varphi) = 0$ and the covariance function is given by*

$$\text{Cov}(W(\varphi_1), W(\varphi_2)) = \text{Cov}(\zeta(\varphi_1), \zeta(\varphi_2)),$$

where $\zeta(\varphi)$ given by

$$\zeta(\varphi) = \left(\frac{\psi(Y)}{C(Y)} - \int_T^Y \frac{\psi(y)}{C^2(y)} dF^*(y), \right)$$

and

$$\psi(w) = \int_{w < t} [\varphi(w) - \varphi(t)] dF(t).$$

In order to prove Theorem 2 we first need two following propositions which their proofs postpone in Section 3.

Proposition 1. *Under Assumption B and condition $J(1) < \infty$, finite dimensional distributions of G_n converges to those of W .*

Remark 1. *Proposition 1 is correct under weak assumptions*

1. $a_G \leq a_F$ and $F\{a_F\} = 0$,
and
2. $\int \frac{dF}{G} < \infty$ and $\int \frac{\varphi^2}{G} dF < \infty$, for every $\varphi \in \mathcal{F}$.

In order to show the uniform CLT, we need to prove that the integral process

$$G_n(\varphi) = \sqrt{n} \int \varphi d(F_n - F) \text{ for } \varphi \in \mathcal{F}$$

is tight in the space of bounded functions acting on the class \mathcal{F}

Proposition 2. *Let $J(1) < \infty$. Then under Assumption **B**, the process $\{G_n(\varphi) : \varphi \in \mathcal{F}\}$ is asymptotically continuous: for every $\epsilon > 0$,*

$$\lim_{\delta \downarrow 0} \limsup_{n \rightarrow \infty} P^* \left\{ \sup_{d(\varphi_1, \varphi_2) < \delta} |G_n(\varphi_1) - G_n(\varphi_2)| > \epsilon \right\} = 0,$$

where

$$d(\varphi_1, \varphi_2) = \left[\int (\varphi_1 - \varphi_2)^2 dF \right]^{1/2}, \text{ for } \varphi_1, \varphi_2 \in \mathcal{F}.$$

Proof of Theorem 2. At first, we notice that from $J(1) < \infty$, one can conclude the total boundedness of the metric space (\mathcal{F}, d) . Now, Proposition 1, Proposition 2 and Pollard [8, Theorem 10.2], complete the proof. \square

3. Proofs

Below we mention some lemmas that are used in the proofs of the main theorems. The proof of the following Lemma 2 and Lemma 3 appear in Stute and Wang [9].

Lemma 2. *Let Assumption **B** is satisfied. Then under condition $J(1) < \infty$, we have*

$$\sqrt{n} \int \varphi d(F_n - F) = n^{-1/2} \sum_{i=1}^n \zeta_i(\varphi) + \sqrt{n} R_n(\varphi).$$

where

$$n^{1/2} R_n(\varphi) = o_p(1).$$

and $\zeta_i(\varphi)$ are i.i.d. copies of the random variables $\zeta(\varphi)$.

Lemma 3. *Under Assumption Lemma 2, we have*

$$\sqrt{n} \int \varphi d(F_n - F_0) \rightarrow N(0, \sigma^2)$$

with

$$\sigma^2 = \text{Var} \left(\frac{\psi(X)}{C(X)} - \int_Y \frac{\psi(y)}{C^2(y)} dF^*(y) \right)$$

Remark 2. Lemma 2 and 3 are correct under weak following assumptions:

1. $a_G \leq a_F$ and $F\{a_F\} = 0$,
2. $\int \frac{dF}{G} < \infty$ and $\int \frac{\varphi^2}{G} dF < \infty$, for every $\varphi \in \mathcal{F}$.

Write $U_n(\varphi) = n^{-1/2} \sum_{i=1}^n \zeta_i(\varphi)$ and $\mathcal{G} = \{\zeta(\varphi) : \varphi \in \mathcal{F}\}$

Proof of Proposition 1. First note that by Lemma 2, one can rewrite $G_n(\varphi)$ as

$$G_n(\varphi) = U_n(\varphi) + \sqrt{n}R_n(\varphi), \quad \text{for } \varphi \in \mathcal{F}.$$

In order to get the one dimensional central limit theorem for G_n , we use Lemma 3 and Slutsky Theorem for each fixed $\varphi \in \mathcal{F}$. the Cramer Wold device implies the result. \square

To prove Proposition 2 we need the following Lemmas.

Lemma 4. Suppose that $J(1) < \infty$, then we have $\int \Phi^2(x) dF(x) < \infty$.

Proof. Similar to the proof of Lemma 1 Bae and Kim [1], from $N_{\square}(1) < \infty$ and

$$\Phi(\cdot) \leq \sum_{i=0}^{N_{\square}(1)} (|l_i(\cdot)| + |u_i(\cdot)|),$$

we can conclude the square integrability of Φ . \square

Lemma 5. Suppose $J(1) < \infty$ and $a_G < a_F$, then

$$\int \frac{dF}{G} < \infty \quad \text{and} \quad \int \frac{\Phi^2}{G} dF < \infty.$$

Proof. From $J(1) < \infty$ and Lemma 4, follows $\int \Phi(x)^2 dF(x) < \infty$. From this fact and $a_G < a_F$, one can write $\int \frac{dF}{G} < \infty$ and $\int \frac{\Phi^2}{G} dF < \infty$. \square

Lemma 6. Under Assumption **B** and condition $J(1) < \infty$, for every $\epsilon > 0$ we have

$$\limsup_{n \rightarrow \infty} P^*(\sqrt{n} \|R_n\|_{\mathcal{F}} > \epsilon) = 0$$

that is $\{\sqrt{n}R_n(\varphi) : \varphi \in \mathcal{F}\}$ is tight.

Proof. From Remark 1.1 of Stute and Wang [9] and Lemma 5, one can write $\|R_n\|_{\mathcal{F}} = o_p(n^{-1/2})$. \square

Lemma 7. Let $J(1) < \infty$, then under Assumption **B**

$$\int_0^1 [\log N_{\square}(\epsilon, \mathcal{G}, d)]^{1/2} d\epsilon < \infty$$

holds, thus Theorem 3.1 and Theorem 3.3. of Ossiander [7] can be applied to the process $\{U_n(\varphi) : \varphi \in \mathcal{F}\}$.

Proof. This result is an easy consequence of Jensen and c_p inequalities. Fix $\epsilon > 0$. By definition of $N_{\square}(\epsilon)$, there exists

$$\left\{ [l_0, u_0], \dots, [l_{N_{\square}(\epsilon)}, u_{N_{\square}(\epsilon)}] \right\}$$

so that for every $\varphi \in \mathcal{F}$ there exists $0 \leq i \leq N_{\square}(\epsilon)$ satisfying $l_i \leq \varphi \leq u_i$ and $d(l_i, u_i) < \epsilon$. Let $g \in \mathcal{G}$. Then $g = \zeta(\varphi)$ for some $\varphi \in \mathcal{F}$. Now define the brackets for the class \mathcal{G} by the equations

$$g_j^l := \frac{\int_{Y < t} (l_j(Y) - u_j(t)) dF(t)}{C(Y)} - \int_T^Y \frac{\int_{y < t} (u_j(y) - l_j(t)) dF(t)}{C^2(y)} dF^*(y)$$

and

$$g_j^u := \frac{\int_{Y < t} (u_j(Y) - l_j(t)) dF(t)}{C(Y)} - \int_T^Y \frac{\int_{y < t} (l_j(y) - u_j(t)) dF(t)}{C^2(y)} dF^*(y),$$

for $j = 0, \dots, N_{\square}(\epsilon)$. Simplify the notations by writing $l = l_j, u = u_j, g^l = g_j^l$ and $g^u = g_j^u$. Obviously, we have $g^l \leq g \leq g^u$. Using c_p inequality and Jensen inequality, we have

$$\begin{aligned} d^2(g^l, g^u) &= \int (g^u - g^l)^2 dF \\ &= \int \left(\int_{Y < t} \frac{[u(Y) - l(Y) + u(t) - l(t)]}{C(Y)} dF(t) \right. \\ &\quad \left. + \int_T^Y \int_{y < t} \frac{[(u(y) - l(y)) + (u(t) - l(t))]}{C^2(y)} dF(t) dF^*(y) \right)^2 dF \\ &\leq C \int \left[\int_{Y < t} \frac{[u(Y) - l(Y) + u(t) - l(t)]}{C(Y)} dF(t) \right]^2 dF \\ &\quad + C \int \left[\int_T^Y \int_{y < t} \frac{[u(y) - l(y) + u(t) - l(t)]}{C^2(y)} dF(t) dF^*(y) \right]^2 dF \\ &\leq C \int \left[\int \frac{(u(Y) - l(Y) + u(t) - l(t))^2}{C^2(Y)} dF(t) \right] dF \\ &\quad + C \int \left[\int_T^Y \frac{u(y) - l(y)}{C^2(y)} dF^*(y) \right. \\ &\quad \left. + \int_T^Y \frac{1}{C^2(y)} \left(\int (u(t) - l(t)) dF(t) \right) dF^*(y) \right]^2 dF \\ &\leq C \int \left[\int \frac{(u(Y) - l(Y) + u(t) - l(t))^2}{C^2(Y)} dF(t) \right] dF \\ &\quad + C \int \left[\int_0^\infty \frac{u(y) - l(y)}{C^2(y)} dF^*(y) \right]^2 dF \end{aligned}$$

$$\begin{aligned}
& + C \int \left[\int_0^\infty \frac{1}{C^2(y)} \left(\int (u(t) - l(t)) dF(t) \right) dF^*(y) \right]^2 dF \\
& \leq C \int \left[\int \frac{(u(Y) - l(Y) + u(t) - l(t))^2}{C^2(Y)} dF(t) \right] dF \\
& + C \int \int_0^\infty \frac{(u(y) - l(y))^2}{C^4(y)} dF^*(y) dF \\
& + C \int \int_0^\infty \frac{1}{C^4(y)} \left(\int (u(t) - l(t)) dF(t) \right)^2 dF^*(y) dp \\
& \leq C \int \left[\int \frac{(u(Y) - l(Y) + u(t) - l(t))^2}{C^2(Y)} dF(t) \right] dF \\
& + C \int \int_0^\infty \frac{(u(y) - l(y))^2}{C^4(y)} dF^*(y) dF \\
& + C \int \int_0^\infty \frac{1}{C^4(y)} \left(\int (u(t) - l(t))^2 dF(t) \right) dF^*(y) dF \\
& =: A_1 + A_2 + A_3.
\end{aligned}$$

The function $C(\cdot)$ is strictly positive on $a_G < y < b_F$, thus

$$\left(\inf_{a_G < y < b_F} C(y) \right)^{-1} > 0.$$

Using c_p inequality, we see that

$$\begin{aligned}
A_1 & \leq C \int \int \frac{(u(Y) - l(Y))^2}{C^2(Y)} dF(t) dF \\
& + C \int \int \frac{(u(t) - l(t))^2}{C^2(Y)} dF(t) dF \\
& \leq C \left(\inf_{a_G < y < b_F} C(y) \right)^{-1} d^2(u, l).
\end{aligned}$$

Also

$$A_2 \leq C \left(\inf_{a_G < y < b_F} C(y) \right)^{-4} d^2(u, l).$$

For the third term C ,

$$A_3 \leq C \left(\inf_{a_G < y < b_F} C(y) \right)^{-4} d^2(u, l).$$

Hence,

$$d^2(g^l, g^u) \leq C d^2(l, u).$$

Finally the result follows from condition $J(1) < \infty$. \square

Proof of Proposition 2. Notice that,

$$|G_n(\varphi_1) - G_n(\varphi_2)| \leq |U_n(\varphi_1) - U_n(\varphi_2)| + 2\sqrt{n}\|R_n\|_{\mathcal{F}} \quad a.s.$$

Now from Lemma 7 and Lemma 6 we can see

$$P^* \{ \|G_n\|_{\delta} > 3\epsilon \} \leq P^* \{ \|U_n\|_{\delta} > \epsilon \} + P^* \left\{ 2n^{1/2} \|R_n\|_{\mathcal{F}} > 2\epsilon \right\} < 3\epsilon.$$

eventually. This complete the proof. \square

REFERENCES

- [1] J. Bae, and S. Kim, The uniform central limit theorem for the Kaplan-Meier integral process, *Bull. Austral. Math. Soc.* **67**(2003), 467–480.
- [2] J. DeHardt, Generalizations of the Glivenko-Cantelli theorem, *Ann. Math. Statist.* **42** (1971), 2050–2055.
- [3] R. M. Dudley, Central limit theorems for empirical measures, *Ann. Probab.* **6**(6)(1978), 899–929.
- [4] S. He and G.L. Yang, The strong law under random truncation, *Ann. Statist.* **26**(1998), 992–1010.
- [5] J. Hoffmann-Jørgensen, *Stochastic processes on Polish spaces* Aarhus Universite, Matematisk Institut, Aarhus. (1991).
- [6] D. Lynden-Bell, A method of allowing for known observational selection in small samples applied to 3CR quasars, *Monthly Notices Roy. Astronom. Soc.* **155**(1971), 95–118.
- [7] M. Ossiander, A central limit theorem under metric entropy with L_2 bracketing, *Ann. Probab.* **15**(1987), 897–919.
- [8] D. Pollard, *Empirical processes: theory and applications*, Regional conference series in Probability and Statistics **2** (Inst. Math. Statist, Hayward CA, 1990).
- [9] W. Stute, and J.-L. Wang, The central limit theorem under random truncation, *Bernoulli*, **14**(3), (2008), 604–622.
- [10] A. Van der Vaart and J. A. Wellner, *Weak Convergence and Empirical Processes*, Springer Series in Statistics, Springer-Verlag, New York. (1996).

E-mail address: fakoor@math.um.ac.ir

E-mail address: rahelehzamini@yahoo.com